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Cut Set Theorems for Rectangular L-fuzzy Complex Numbers and L-multi-fuzzy Complex Numbers

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Abstract

In this paper, the properties of the lower cut set and the upper cut set of rectangular L-fuzzy complex numbers are studied and it is showed that the (ℓ)-upper cut and $[\ell]$-upper cut on this type of fuzzy numbers are satisfied under β and α preserving mapping, respectively. The inclusion properties on the set of cut set types are proved and some decompositions of this type of number and its characterizations are discussed. The basic arithmetic operations between rectangular L-fuzzy complex numbers are introduced and some representations of them are obtained. Furthermore, the notions of rectangular L-multi-fuzzy complex numbers are demonstrated and some fundamental theorems and rules were presented for calculating binary operations between them. Some modulus and inequalities on the set of L-multi-fuzzy quantities are derived.

Introduction

Since previous research on fuzzy numbers and fuzzy complex numbers have been based on $[0,1]$-fuzzy analysis theory, their applications are limited. Therefore, based on Zadeh's work [1], Goguen [2] generalized fuzzy sets to the notion of L-fuzzy sets where L is a complete lattice. Later, Huang and Shi [3] generalized the notions of fuzzy convex sets, fuzzy numbers, and cardinality of fuzzy sets to the L-fuzzy systems theory. They presented some of their equivalent properties and characterizations in terms of cut sets of the theory of L-fuzzy sets as an application similar to [4].

Multi fuzzy set theory is an extension of theories of intuitionistic fuzzy sets [5] and L-fuzzy sets. Fuzzy multisets were the first introduced by Yager [6] which discussed the basic operations and properties. Next, Miyamoto [7] modified and developed some new operations that recover Yager's multisets. Moreover, to generalize fuzzy numbers to the general setting, Sabir [8] introduced and studied multi-fuzzy numbers as an extension of fuzzy numbers in [9]. Also, Dey and Pal [10] defined multi-fuzzy complex numbers and sets by investigating the ordered sequences of a grade of memberships.

The paper is organized as follows: In section two, we review the definitions and concepts related to the topic and present the notations needed in the rest of the paper. In section three, we present some important results involving rectangular L-fuzzy complex numbers and prove some new results and connections among them when L is a completely distributive lattice. In section four, we consider L-multi-fuzzy complex numbers with finite membership sequences and their characterization using set-valued membership functions.

Preliminaries

Zadeh [1] presented the concept of fuzzy sets from crisp sets and defined fuzzy set $\tilde{x}$ on the universal set X, which is a mapping $\mu(x|\tilde{x}): X \to [0,1]$. Frequently, we shall denote the collection of all fuzzy sets in X by $\mathcal{F}(\mathrm{X})$ and sometimes write $\mu_{\tilde{x}}(x)$ instead of $\mu(x|\tilde{x})$. Let $\tilde{x}_i \in \mathcal{F}(X)$, i in a nonempty index set I. Then the standard fuzzy intersection of $\tilde{x}_i$, $\tilde{\cap}_i \tilde{x}_i$, and the standard fuzzy union of $\tilde{x}_i$, $\tilde{\cup}_i \tilde{x}_i$, are defined, respectively, by $\inf_x \mu(x|\tilde{x}_i) := \bigwedge_x \mu_{\tilde{x}_i}(x)$ and $\sup_x \mu(x|\tilde{x}_i) := \bigvee_x \mu_{\tilde{x}_i}(x)$.

One of the basic notions of fuzzy sets is the Zadeh's extension principle which provides an important method for extending regular mathematical notions to fuzzy quantities as the arguments of the function. Let $f: A_1 \times A_2 \times \ldots \times A_n \to B$ given by $x = f(x_1, x_2, \ldots, x_n)$ and $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ are n fuzzy sets on X_i for $i = 1,2, \ldots n$. Here the extension principal set $\tilde{x} = f(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is defined by

$$\mu(x|\tilde{x}) = \bigvee_{x_1, x_2, \ldots, x_n \mid x = f(x_1, x_2, \ldots, x_n)} (\mu(x_1|\tilde{x}_1) \wedge \mu(x_2|\tilde{x}_2) \wedge \ldots \wedge \mu(x_n|\tilde{x}_n)).$$

The fuzzy control system is a generalization of the human experience of formulating control actions using linguistic rules with ambiguous predicates. The key to employing linguistic rules, both in the premise and in the consequent part of the rules, is a well-established use of t-norms and t-conorms for modeling the intersection and the union of fuzzy sets. A triangular norm is a binary operation T on the unit interval satisfying commutativity, associativity, monotonicity, and boundary conditions. The basic triangular norms [11] are $T_m(a,b) = \wedge(a,b)$, $T_b(a,b) = \vee(0, a+b-1)$ and $T_p(a,b) = ab$.

A partially ordered set L is called a lattice if for any finite subset has both supremum and infimum in L. A lattice L is said to be completely distributive [12] if the following conditions hold

(1) $\bigwedge_{i\in I}\left(\bigvee_{j\in J_i} a_{i,j}\right) = \bigvee_{f\in\prod J_i}\left(\bigwedge_{i\in I} a_{i,f(i)}\right)$,

(2) $\bigvee_{i\in I}\left(\bigwedge_{j\in J_i} a_{i,j}\right) = \bigwedge_{f\in\prod J_i}\left(\bigvee_{j\in J_i} a_{i,f(i)}\right)$,

where for all $i \in I$ and for all $j \in J_i$, $a_{i,j} \in L$ and $f \in \prod J_i$ means f is a mapping $f: I \to \bigcup J_i$ such that for all $i \in I, f(i) \in J_i$.

An element a in a lattice L is called prime element [13] if and only if the relation $x \wedge y \leq a$ always implies $x \leq a$ or $y \leq a$. An element is co-prime if and only if it is a prime of L^{op} as short for $(L, \leq^{op})$. The set of non-unit prime elements (resp. non-zero co-prime elements) is denoted by $P(L)$ (resp. $M(L)$).

Wang in [14] presented some definitions and results as follows:

Definition 2.1. Let L be a complete lattice, $a \in L$. Then $\phi \neq B \subset L$ is called a minimal family of a if the following conditions hold

(1) $\sup(B) = a$,

(2) for all $A \subset L$, $\sup A \leq a$ implies that for all $x \in B$ there exists $y \in A$ such that $y \geq x$.

Definition 2.2. Let L be a complete lattice, $a \in L$. Then $\phi \neq A \subset L$ is called a maximal family of a if the following conditions are satisfied

(1) $\inf A = a$,

(2) for all $B \subset L$, $\inf B \leq a$ implies that for all $x \in A$ there exists $y \in B$ such that $y \leq x$.

Through this paper, $\beta(a)$ will denote the greatest minimal family of a; and $\alpha(a)$ will denote the greatest maximal family of a, if it exists. For each $a \in L$, there exists $\alpha(a)$ and $\beta(a)$ such that $a = \bigwedge \alpha(a) = \bigvee \beta(a)$.

Theorem 2.3. Let L be a completely distributive lattice, $a, b \in L$ and $a \geq b$ (resp. $a \leq b$). Then $\beta(b) \subset \beta(a)$ (resp. $\alpha(b) \subset \alpha(a)$).

Theorem 2.4. Let L be a completely distributive lattice, $a_i \in L$ for all $i \in I$. Then

(1) $\beta(\bigvee_i a_i) = \bigcup_i \beta(a_i)$, that is, β is a union-preserving mapping.

(2) $\alpha(\bigwedge_{i\in I} a_i) = \bigcup_{i\in I} \alpha(a_i)$, that is, α is a $\bigwedge - \bigvee$ mapping.

Noted that α need not be a $\bigvee - \bigwedge$ mapping. For example, let $L = [0,1]$ and $a_i = \frac{1}{2} - \frac{1}{10i}$, then $\bigvee_i a_i = \frac{1}{2}$ and $\alpha(a_i) = \left(\frac{1}{2} - \frac{1}{10i}, 1\right]$. Hence, $\alpha(\bigvee_i a_i) = \left(\frac{1}{2}, 1\right] \neq \left[\frac{1}{2}, 1\right] = \bigcap_{i\in I} \alpha(a_i)$.

As a generalization of Zadeh's fuzzy set, the concept of L-fuzzy set [2] was presented that map whole members in universal set X to L. Throughout this paper, $L(X)$ denotes the set of all L-fuzzy sets (briefly L-sets) over non-empty set X. Let $\tilde{x} \in L(X)$ and $a \in L$, cut sets [3,4,15] of $\tilde{x}$ are defined as $\tilde{x}_{[a]} = \{x \in X : \mu(x|\tilde{x}) \geq a\}$, $\tilde{x}_{(a)} = \{x \in X : a \in \beta(\mu(x|\tilde{x}))\}$, $\tilde{x}^{[a]} = \{x \in X : a \notin \alpha(\mu(x|\tilde{x}))\}$ and $\tilde{x}^{(a)} = \{x \in X : \mu(x|\tilde{x}) \nleq a\}$. It is obvious that $a \in \beta(b)$ implies $\tilde{x}_{[b]} \subset \tilde{x}_{[a]} \subset \tilde{x}_{[a]}$ and $a \in \alpha(b)$ implies $\tilde{x}^{[a]} \subset \tilde{x}^{(a)} \subset \tilde{x}^{[a].}$

Let $\mathbb{R}$ denote the set of real numbers. Huang and Shi in [3] defined that $\tilde{x} \in L(\mathbb{R})$ is an L-fuzzy number if there exists $x \in \mathbb{R}$ such that $\mu(x|\tilde{x}) = 1$ (that is, $\tilde{x}$ is normal) and for all $a \in L$, $\tilde{x}_{[a]}$ is a closed interval. Next, the author in [16] redefine the L-fuzzy number as follows:

Definition 2.5. An L-set $\tilde{x}$ of $\mathbb{R}$ is called an L-fuzzy number if

(1) $\tilde{x}$ is normal,

(2) for all $0 \neq a \in L$, $\tilde{x}_{[a]}$ is a bounded closed interval.

Theorem 2.6. If $\tilde{x}$ and $\tilde{y}$ are real L-fuzzy numbers, then $\tilde{x} + \tilde{y}$, $\tilde{x} - \tilde{y}$, $\tilde{x} \cdot \tilde{y}$ are also L-fuzzy number; If $\tilde{y}$ is a positive L-fuzzy number (or a negative L-fuzzy number) then $\tilde{x} \div \tilde{y}$ is also L-fuzzy number.

Rectangular L-fuzzy complex numbers and their cut sets

In this section, we present and discuss some characterizations and properties of rectangular L-fuzzy complex numbers and their cut sets. Also, we concentrate on the modulus and conjugate of rectangular L-fuzzy complex numbers.

Definition 3.1. If $\tilde{x}$ and $\tilde{y}$ are two L-fuzzy numbers with membership functions $\mu(x|\tilde{x})$ and $\mu(y|\tilde{y})$, respectively, then $\tilde{z} = \tilde{x} + i\tilde{y}$ is a rectangular L-fuzzy complex number with membership function

$$\mu(x + iy|\tilde{z}) = T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big).$$

We will use $RLF(\mathbb{C})$ to denote the set of all rectangular L-fuzzy complex numbers.

Definition 3.2. Let $\tilde{z} \in RLF(\mathbb{C})$ and $\ell \in L$. We define, the $[\ell]$-upper cut set of $\tilde{z}$, $\tilde{z}^{[\ell]} = \{z \in \mathbb{C} \mid \mu(x + iy|\tilde{z}) \geq \ell\}$, the (ℓ)-upper cut set of $\tilde{z}$, $\tilde{z}^{(\ell)} = \{z \in \mathbb{C} \mid \ell \in \beta(\mu(x + iy|\tilde{z}))\}$, the (ℓ)-lower cut set of $\tilde{z}$, $\tilde{z}_{(\ell)} = \{z \in \mathbb{C} \mid \mu(x + iy|\tilde{z}) > \ell\}$, the $[\ell]$-lower cut set of $\tilde{z}$, $\tilde{z}_{[\ell]} = \big\{z \in \mathbb{C} \mid \ell \notin \alpha\big(\mu(x + iy|\tilde{z})\big)\big\}$, respectively.

Definition 3.3. If $\tilde{z}_1, \tilde{z}_2 \in RLF(\mathbb{C})$ and $* \in \{+, -, \cdot, \div\}$, then we define the membership function of $\tilde{z}_1 * \tilde{z}_2$ as follows

$$\mu(w \mid \tilde{z}_1 * \tilde{z}_2) = \bigvee_{w = w_1 * w_2} T_m\big(\mu(w_1|\tilde{z}_1), \mu(w_2|\tilde{z}_2)\big)$$

Lemma 3.4. Let $\tilde{z} \in RLF(\mathbb{C})$ and $\mu \in L$. Then

(1) $\tilde{z}^{[\mu]} \subset \tilde{z}^{(\ell)} \subset \tilde{z}^{[\ell]}$ for all $\ell \in \beta(\mu)$.
(2) $\tilde{z}_{[\ell]} \subset \tilde{z}_{(\mu)} \subset \tilde{z}_{[\mu]}$ for all $\ell \in \alpha(\mu)$.

Proof. We prove only the first part and the second part is similar. Let $z \in \tilde{z}^{[\mu]}$, for all $\ell \in \beta(\mu)$, implies that $\mu(z|\tilde{z}) \geq \mu$, from Theorem 2.3 we obtain that $\beta(\mu) \subset \beta\big(\mu(z|\tilde{z})\big)$ and this means that $\ell \in \beta\big(\mu(z|\tilde{z})\big)$. Therefore, $z \in \tilde{z}^{(\ell)}$.

In order to prove $\tilde{z}^{(\ell)} \subset \tilde{z}^{[\ell]}$, let $z \in \tilde{z}^{(\ell)}$ then $\ell \in \beta\big(\mu(z|\tilde{z})\big)$ implies that $\ell \leq \sup\Big(\beta\big(\mu(z|\tilde{z})\big)\Big)$. So, by Definition 2.1, $\ell \leq \sup\Big(\beta\big(\mu(z|\tilde{z})\big)\Big) = \mu(z|\tilde{z})$.

Lemma 3.5. Let $\tilde{z} \in RLF(\mathbb{C})$ and $\ell \in L$, we have

(1) $\tilde{z}^{[\ell]} = \tilde{x}^{[\ell]} + i\,\tilde{y}^{[\ell]}$.
(2) $\tilde{z}_{[\ell]} = \tilde{x}_{[\ell]} + i\,\tilde{y}_{[\ell]}$.
(3) $\tilde{z}_{(\ell)} = \tilde{x}_{(\ell)} + i\,\tilde{y}_{(\ell)}$.

Proof. (1) Suppose that z belongs to the $[\ell]$-upper cut of $\tilde{z}$, then there exists x and y so that $x + iy = z$ and $T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big) \geq \ell$. This implies that, both $\mu(x|\tilde{x})$ and $\mu(y|\tilde{y})$ are greater than or equals ℓ. Therefore, by definition of ℓ-cut of L-fuzzy numbers, $x \in \tilde{x}^{[\ell]}$, $y \in \tilde{y}^{[\ell]}$ and $z \in \tilde{x}^{[\ell]} + i\,\tilde{y}^{[\ell]}$. Conversely, if $z \in \tilde{x}^{[\ell]} + i\,\tilde{y}^{[\ell]}$, where $z = x + iy$, then obviously, $T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big) \geq \ell$ and $z \in \tilde{Z}^{[\ell]}$.

(2) Let z belongs to the $[\ell]$-lower cut of $\tilde{z}$, then there is an x and y such that $x+iy=z$ and $\ell \notin \alpha\left(T_m\big(\mu(x|\tilde{x}),\mu(y|\tilde{y})\big)\right)$. So, by Theorem 2.4 we get that $\alpha\big(\mu(x|\tilde{x}) \wedge \mu(y|\tilde{y})\big) = \alpha\big(\mu(x|\tilde{x}\,)\big) \cup \alpha\big(\mu(y|\tilde{y}\,)\big)$. This is to say $\ell \notin \alpha\big(\mu(x|\tilde{x}\,)\big)$ and $\ell \notin \alpha\big(\mu(y|\tilde{y}\,)\big)$. So that, by definition of ℓ-cut of L-fuzzy numbers, $x \in \tilde{x}_{[\ell]}$, $y \in \tilde{y}_{[\ell]}$ and $z \in \tilde{x}_{[\ell]} + i\,\tilde{y}_{[\ell]}$. Conversely, if $z \in \tilde{x}_{[\ell]} + i\,\tilde{y}_{[\ell]}$, where $z = x+iy$, then obviously, $\ell \notin \alpha\big(\mu(x|\tilde{x}\,)\big) \cup \alpha\big(\mu(y|\tilde{y}\,)\big) = \alpha\left(T_m\big(\mu(x|\tilde{x}),\mu(y|\tilde{y})\big)\right)$, and $z \in \tilde{z}_{[\ell]}$.

(3) The proof is similar to that of (1)

Lemma 3.6. Let $\tilde{z}$ be a rectangular L-fuzzy complex number. If for any $\mu_1,\mu_2 \in L$, $\beta(\mu_1 \wedge \mu_2) = \beta(\mu_1) \cap \beta(\mu_2)$, then $\tilde{z}^{(\ell)} = \tilde{x}^{(\ell)} + i\tilde{y}^{(\ell)}$ for all $\ell \in L$.

Proof. Let $z \in \tilde{z}^{(\ell)}$, then there is an x and y so that $x+iy=z$ and $\ell \in \beta\left(T_m\big(\mu(x|\tilde{x}),\mu(y|\tilde{y})\big)\right)$. Because $\beta(\ell \wedge \mu) = \beta(\ell) \cap \beta(\mu), \forall \ell,\mu \in L$, so it yields that

$$\beta\left(T_m\big(\mu(x|\tilde{x}),\mu(y|\tilde{y})\big)\right) = \beta\big(\mu(x|\tilde{x}\,)\big) \cap \beta\big(\mu(y|\tilde{y}\,)\big).$$

This implies that $\ell \in \beta\big(\mu(x|\tilde{x}\,)\big)$ and $\ell \in \beta\big(\mu(y|\tilde{y}\,)\big)$. Therefore, by definition of ℓ-cut of L-fuzzy numbers, $x \in \tilde{x}^{(\ell)}$, $y \in \tilde{y}^{(\ell)}$ and $z \in \tilde{x}^{(\ell)} + i\,\tilde{y}^{(\ell)}$.

Conversely, if $z \in \tilde{x}^{(\ell)} + i\,\tilde{y}^{(\ell)}$, where $z = x+iy$, then obviously, $\ell \in \beta\left(T_m\big(\mu(x|\tilde{x}),\mu(y|\tilde{y})\big)\right)$ and $z \in \tilde{z}^{(\ell)}$ straight forward.

Theorem 3.7. Let $\tilde{z}_1,\tilde{z}_2 \in RLF(\mathbb{C})$ and $\ell \in L$. Then

(1) $(\tilde{z}_1+\tilde{z}_2)^{(\ell)} \subset \tilde{z}_1{}^{(\ell)} + \tilde{z}_2{}^{(\ell)} \subset \tilde{z}_1{}^{[\ell]} + \tilde{z}_2{}^{[\ell]} \subset (\tilde{z}_1+\tilde{z}_2)^{[\ell]}$.
(2) $(\tilde{z}_1+\tilde{z}_2)_{(\ell)} \subset \tilde{z}_{1_{(\ell)}} + \tilde{z}_{2_{(\ell)}} \subset \tilde{z}_{1_{[\ell]}} + \tilde{z}_{2_{[\ell]}} \subset (\tilde{z}_1+\tilde{z}_2)_{[\ell]}$.

Proof. (1) Let $w \in (\tilde{z}_1+\tilde{z}_2)^{(\ell)}$ implies that $\ell \in \beta\big(\mu(w|\tilde{z}_1+\tilde{z}_2)\big)$, From Definition 3.3, we obtain that

$$\ell \in \beta\left(\bigvee_{w=w_1+w_2} T_m\big(\mu(w_1|\tilde{z}_1),\mu(w_2|\tilde{z}_2)\big)\right)$$

and by Theorem 2.4, we have

$$\ell \in \bigcup_{w=w_1+w_2} \beta\left(T_m\big(\mu(w_1|\tilde{z}_1),\mu(w_2|\tilde{z}_2)\big)\right).$$

Hence, there exists w_1 and w_2 satisfying $w_1+w_2=w$ such that $\ell \in \beta\left(T_m\big(\mu(w_1|\tilde{z}_1),\mu(w_2|\tilde{z}_2)\big)\right)$. This implies that ℓ belongs to both $\beta\big(\mu(w_1|\tilde{z}_1)\big)$ and $\big(\mu(w_2|\tilde{z}_2)\big)$. For $w_1 = x_1+iy_1$ and $w_2 = x_2+iy_2$, Definition 3.1 say that ℓ belongs to all $\beta\big(\mu(x_1|\tilde{x}_1)\big)$, $\beta\big(\mu(x_2|\tilde{x}_2)\big)$, $\beta\big(\mu(y_1|\tilde{y}_1)\big)$ and $\beta\big(\mu(y_2|\tilde{y}_2)\big)$. Therefore, the definition of ℓ-cut of L-fuzzy numbers implies that $w_1 \in \tilde{x}_1{}^{(\ell)} + i\tilde{y}_1{}^{(\ell)}$ and $w_2 \in \tilde{x}_2{}^{(\ell)} + i\tilde{y}_2{}^{(\ell)}$. Hence, by Lemma 3.6 we get that $w \in \tilde{z}_1{}^{(\ell)} + \tilde{z}_2{}^{(\ell)}$.

On the other hand, by Lemma 3.4 we have ${\tilde{z}_1}^{(\ell)} \subset {\tilde{z}_1}^{[\ell]}$ and ${\tilde{z}_2}^{(\ell)} \subset {\tilde{z}_2}^{[\ell]}$, this means that ${\tilde{z}_1}^{(\ell)} + {\tilde{z}_2}^{(\ell)} \subset {\tilde{z}_1}^{[\ell]} + {\tilde{z}_2}^{[\ell]}$.

To prove that ${\tilde{z}_1}^{[\ell]} + {\tilde{z}_2}^{[\ell]} \subset (\tilde{z}_1 + \tilde{z}_2)^{[\ell]}$, let $w \in {\tilde{z}_1}^{[\ell]} + {\tilde{z}_2}^{[\ell]}$, then there exists w_1 and w_2 satisfying $w_1 + w_2 = w$ such that $w_1 \in {\tilde{z}_1}^{[\ell]}$ and $w_2 \in {\tilde{z}_2}^{[\ell]}$, which implies that $\mu(w_1|\tilde{z}_1) \geq \ell$ and $\mu(w_2|\tilde{z}_2) \geq \ell$. For $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, by Definition 3.1 we get that $T_m\big(\mu(x_1|\tilde{x}_1), \mu(y_1|\tilde{y}_1)\big) \geq \ell$ and $T_m\big(\mu(x_2|\tilde{x}_2), \mu(y_2|\tilde{y}_2)\big) \geq \ell$

In other words, $T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big) \geq \ell$ and $T_m\big(\mu(y_1|\tilde{y}_1), \mu(y_2|\tilde{y}_2)\big) \geq \ell$. Next, the extension principle for L-fuzzy numbers, obtained that

$$\mu(x|\tilde{x}_1 + \tilde{x}_2) = \bigvee_{x = x_1 + x_2} T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big) \geq \ell,$$

and,

$$\mu(y|\tilde{y}_1 + \tilde{y}_2) = \bigvee_{y = y_1 + y_2} T_m\big(\mu(y_1|\tilde{y}_1), \mu(y_2|\tilde{y}_2)\big) \geq \ell.$$

Therefore,

$$x \in (\tilde{x}_1 + \tilde{x}_2)^{[\ell]} \text{ and } y \in (\tilde{y}_1 + \tilde{y}_2)^{[\ell]}.$$

Finally, Lemma 3.5 (1) implies that, $w \in (\tilde{z}_1 + \tilde{z}_2)^{[\ell]}$.

The proof of (2) is similar as (1).

Theorem 3.8. For the rectangular L-fuzzy complex numbers $\tilde{z}_1$ and $\tilde{z}_2$ and $\ell \in L$, we have

(1) $(\tilde{z}_1 + \tilde{z}_2)^{[\ell]} = \cap_{\mu \in \beta(\ell)} \big({\tilde{z}_1}^{[\mu]} + {\tilde{z}_2}^{[\mu]}\big) = \cap_{\mu \in \beta(\ell)} \big({\tilde{z}_1}^{(\mu)} + {\tilde{z}_2}^{(\mu)}\big)$.

(2) $(\tilde{z}_1 + \tilde{z}_2)^{(\ell)} = \cup_{\ell \in \beta(\mu)} \big({\tilde{z}_1}^{[\mu]} + {\tilde{z}_2}^{[\mu]}\big) = \cup_{\ell \in \beta(\mu)} \big({\tilde{z}_1}^{(\mu)} + {\tilde{z}_2}^{(\mu)}\big)$.

(3) $(\tilde{z}_1 + \tilde{z}_2)_{[\ell]} = \cap_{\ell \in \alpha(\mu)} \left(\tilde{z}_{1[\mu]} + \tilde{z}_{2[\mu]}\right) = \cap_{\ell \in \alpha(\mu)} \left(\tilde{z}_{1(\mu)} + \tilde{z}_{2(\mu)}\right)$.

(4) $(\tilde{z}_1 + \tilde{z}_2)_{(\ell)} = \cup \left(\tilde{z}_{1[\mu]} + \tilde{z}_{2[\mu]}\right)_{\mu \in \alpha(\ell)} = \cup \left(\tilde{z}_{1(\mu)} + \tilde{z}_{2(\mu)}\right)_{\mu \in \alpha(\ell)}$.

Proof. (1) Let $w \in (\tilde{z}_1 + \tilde{z}_2)^{[\ell]}$ implies $\mu(w|\tilde{z}_1 + \tilde{z}_2) \geq \ell$. Definition 3.3 gets that,

$$\bigvee_{w = w_1 + w_2} T_m\big(\mu(w_1|\tilde{z}_1), \mu(w_2|\tilde{z}_2)\big) \geq \ell$$

Hence, there exists w_1 and w_2 satisfying $w_1 + w_2 = w$ such that $T_m\big(\mu(w_1|\tilde{z}_1), \mu(w_2|\tilde{z}_2)\big) \geq \ell$. This implies, $\mu(w_1|\tilde{z}_1) \geq \ell$ and $\mu(w_2|\tilde{z}_2) \geq \ell$. For $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, by Definition 3.1 we get that

$$T_m\big(\mu(x_1|\tilde{x}_1), \mu(y_1|\tilde{y}_1)\big) \geq \ell \text{ and } T_m\big(\mu(x_2|\tilde{x}_2), \mu(y_2|\tilde{y}_2)\big) \geq \ell$$

In another word, $\mu(x_1|\tilde{x}_1) \geq \ell$, $\mu(x_2|\tilde{x}_2) \geq \ell$, $\mu(y_1|\tilde{y}_1) \geq \ell$, and $\mu(y_2|\tilde{y}_2) \geq \ell$.

Therefore, by noting that ${\tilde{x}_1}^{[\ell]} \subset {\tilde{x}_1}^{[\mu]}$ and ${\tilde{x}_2}^{[\ell]} \subset {\tilde{x}_2}^{[\mu]}$ for each $\mu \in \beta(\ell)$, we can obtain ${\tilde{x}_1}^{[\ell]} \subset \bigcap_{\mu\in\beta(\ell)} {\tilde{x}_1}^{[\mu]}$ and ${\tilde{x}_2}^{[\ell]} \subset \bigcap_{\mu\in\beta(\ell)} {\tilde{x}_2}^{[\mu]}$ and this implies that

$$x_1 \in \bigcap_{\mu\in\beta(\ell)} {\tilde{x}_1}^{[\mu]} \text{ and } x_2 \in \bigcap_{\mu\in\beta(\ell)} {\tilde{x}_2}^{[\mu]}.$$

As before, we can also have

$$y_1 \in \bigcap_{\mu\in\beta(\ell)} {\tilde{y}_1}^{[\mu]} \text{ and } y_2 \in \bigcap_{\mu\in\beta(\ell)} {\tilde{y}_2}^{[\mu]}.$$

This means,

$$w_1 \in \bigcap_{\mu\in\beta(\ell)} \left({\tilde{x}_1}^{[\mu]} + i\,{\tilde{y}_1}^{[\mu]}\right) \text{ and } w_2 \in \bigcap_{\mu\in\beta(\ell)} \left({\tilde{x}_2}^{[\mu]} + i\,{\tilde{y}_2}^{[\mu]}\right)$$

Hence, by Lemma 3.5 (1) we get that

$$w \in \bigcap_{\mu\in\beta(\ell)} \left({\tilde{x}_1}^{[\mu]} + i\,{\tilde{y}_1}^{[\mu]} + {\tilde{x}_2}^{[\mu]} + i\,{\tilde{y}_2}^{[\mu]}\right) = \bigcap_{\mu\in\beta(\ell)} \left({\tilde{z}_1}^{[\mu]} + {\tilde{z}_2}^{[\mu]}\right).$$

Conversely, if we take

$$w \in \bigcap_{\mu\in\beta(\ell)} \left({\tilde{z}_1}^{[\mu]} + {\tilde{z}_2}^{[\mu]}\right),$$

then for all $\mu \in \beta(\ell)$, we get $w \in \left({\tilde{z}_1}^{[\mu]} + {\tilde{z}_2}^{[\mu]}\right)$. This implies, there exists w_1 and w_2 such that $w_1 + w_2 = w$ and both belong to ${\tilde{z}_1}^{[\mu]}$ and ${\tilde{z}_2}^{[\mu]}$, respectively.

So that, for $w_1 = x_1 + iy_1, w_2 = x_2 + iy_2$ and all $\mu \in \beta(\ell)$, Definition 3.1 and Definition 3.2 implies that both $T_m\big(\mu(x_1|\tilde{x}_1), \mu(y_1|\tilde{y}_1)\big)$ and $T_m\big(\mu(x_2|\tilde{x}_2), \mu(y_2|\tilde{y}_2)\big)$ are greater than or equal to μ.

Next, the extension principle for L-fuzzy numbers obtained that,

$$\mu(x|\tilde{x}_1 + \tilde{x}_2) = \bigvee_{x=x_1+x_2} T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big) \geq \bigvee\{\mu \in L \mid \mu \in \beta(\ell)\} = \ell,$$

and,

$$\mu(y|\tilde{y}_1 + \tilde{y}_2) = \bigvee_{y=y_1+y_2} T_m\big(\mu(y_1|\tilde{y}_1), \mu(y_2|\tilde{y}_2)\big) \geq \ell.$$

Therefore,

$$x \in (\tilde{x}_1 + \tilde{x}_2)^{[\ell]} \text{ and } y \in (\tilde{y}_1 + \tilde{y}_2)^{[\ell]}.$$

Finally, Lemma 3.5 (1) implies that, $w = x + iy \in (\tilde{z}_1 + \tilde{z}_2)^{[\ell]}$.

To prove $(\tilde{z}_1+\tilde{z}_2)^{[\ell]} = \bigcap_{\mu\in\beta(\ell)}(\tilde{z}_1{}^{(\mu)} + \tilde{z}_2{}^{(\mu)})$, we take $w \in \bigcap_{\mu\in\beta(\ell)}(\tilde{z}_1{}^{(\mu)} + \tilde{z}_2{}^{(\mu)})$. Then there exists w_1 and w_2 such that $w_1+w_2=w$ and μ belongs to both $\beta\left(T_m\big(\mu(x_1|\tilde{x}_1),\mu(y_1|\tilde{y}_1)\big)\right)$ and $\beta\left(T_m\big(\mu(x_2|\tilde{x}_2),\mu(y_2|\tilde{y}_2)\big)\right)$, this implies that $\beta(\ell)\subset\beta\left(T_m\big(\mu(x_1|\tilde{x}_1),\mu(y_1|\tilde{y}_1)\big)\right)$ and $\beta(\ell)\subset\beta\left(T_m\big(\mu(x_2|\tilde{x}_2),\mu(y_2|\tilde{y}_2)\big)\right)$, by Theorem 2.3 we obtain that $T_m\big(\mu(x_1|\tilde{x}_1),\mu(y_1|\tilde{y}_1)\big)\geq\ell$ and $T_m\big(\mu(x_2|\tilde{x}_2),\mu(y_2|\tilde{y}_2)\big)\geq\ell$. Next, the Zadeh's extension for L-fuzzy numbers get that,

$$\mu(x|\tilde{x}_1+\tilde{x}_2) = \bigvee_{x=x_1+x_2} T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2)\big)\geq\ell,$$

and

$$\mu(y|\tilde{y}_1+\tilde{y}_2) = \bigvee_{y=y_1+y_2} T_m\big(\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)\geq\ell.$$

Therefore,

$$x\in(\tilde{x}_1+\tilde{x}_2)^{[\ell]} \text{ and } y\in(\tilde{y}_1+\tilde{y}_2)^{[\ell]}$$

Hence, Lemma 3.5 (1) implies that, $w=x+iy\in(\tilde{z}_1+\tilde{z}_2)^{[\ell]}$.

The proof of other inclusion obtained by the same way.

The proof of (2), (3) and (4) are similar steps.

Theorem 3.9. If $\tilde{z}_1,\tilde{z}_2\in RLF(\mathbb{C})$ and $*\in\{+,-,\cdot,\div\}$, then

(1) $\tilde{z}_1*\tilde{z}_2 = \bigvee_{\ell\in L}\ (\ell\wedge(\tilde{z}_1{}^{[\ell]}*\tilde{z}_2{}^{[\ell]})) = \bigvee_{\ell\in L}\ (\ell\wedge(\tilde{z}_1{}^{(\ell)}*\tilde{z}_2{}^{(\ell)})) = \bigvee_{\ell\in M(L)}\ (\ell\wedge(\tilde{z}_1{}^{[\ell]}*\tilde{z}_2{}^{[\ell]})) = \bigvee_{\ell\in M(L)}\ (\ell\wedge(\tilde{z}_1{}^{(\ell)}*\tilde{z}_2{}^{(\ell)}))$.

(2) $\tilde{z}_1*\tilde{z}_2 = \bigwedge_{\ell\in L}(\ell\vee(\tilde{z}_{1[\ell]}*\tilde{z}_{2[\ell]})) = \bigwedge_{\ell\in L}(\ell\vee(\tilde{z}_{1(\ell)}*\tilde{z}_{2(\ell)})) = \bigwedge_{\ell\in P(L)}(\ell\vee(\tilde{z}_{1[\ell]}*\tilde{z}_{2[\ell]})) = \bigwedge_{\ell\in P(L)}(\ell\vee(\tilde{z}_{1(\ell)}*\tilde{z}_{2(\ell)}))$.

Proof. We prove only for the case when $*$ is addition. Let $\tilde{z}_1=\tilde{x}_1+i\tilde{y}_1$ and $\tilde{z}_2=\tilde{x}_2+i\tilde{y}_2$, first we have to prove that $\tilde{z}_1+\tilde{z}_2=\tilde{x}_1+\tilde{x}_2+i(\tilde{y}_1+\tilde{y}_2)$. We know that for $w_1=x_1+iy_1$ and $w_2=x_2+iy_2$, $T_m\big(\mu(w_1|\tilde{z}_1),\mu(w_2|\tilde{z}_2)\big)=T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2),\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)$. Furthermore, let $\tilde{x}$, $\tilde{y}$, x, and y denotes for $\tilde{x}_1+\tilde{x}_2$, $\tilde{y}_1+\tilde{y}_2$, x_1+x_2, and y_1+y_2 respectively, then

$$T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2),\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)\leq T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2)\big)$$

and

$$T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2),\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)\leq T_m\big(\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)$$

This means that,

$T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2),\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)$ less than or equal to both $\mu(x\,|\tilde{x})$ and $\mu(y\,|\tilde{y})$.

Hence, for $\tilde{z}=\tilde{x}+i\tilde{y}$ and $w=x=iy$, $T_m\big(\mu(x_1|\tilde{x}_1),\mu(x_2|\tilde{x}_2),\mu(y_1|\tilde{y}_1),\mu(y_2|\tilde{y}_2)\big)\leq\mu(w\,|\tilde{z})$ implies that $\mu(w\mid\tilde{z}_1+\tilde{z}_2)\leq\mu(z\,|\tilde{z})$.

On the other hand, for arbitrary $\epsilon > 0$ there are x_1', y_1', x_2' and y_2' such that $x = x_1' + x_2'$, $y = y_1' + y_2'$, and $T_m\big(\mu(x_1'|\tilde{x}_1), \mu(x_2'|\tilde{x}_2)\big) > \mu(x\,|\tilde{x}) - \epsilon$ $\big(Resp. T_m\big(\mu(y_1'|\tilde{y}_1), \mu(y_2'|\tilde{y}_2)\big) > \mu(y\,|\tilde{y}) - \epsilon\big)$. This implies that,

$$T_m\big(\mu(x_1'|\tilde{x}_1), \mu(x_2'|\tilde{x}_2), \mu(y_1'|\tilde{y}_1), \mu(y_2'|\tilde{y}_2)\big) > \mu(w\,|\tilde{z}) - \epsilon$$

Therefore, $\mu(w \mid \tilde{z}_1 + \tilde{z}_2) > \mu(w\,|\tilde{z}) - \epsilon$ implies that $\mu(w \mid \tilde{z}_1 + \tilde{z}_2) = \mu(w\,|\tilde{z})$.

Finally, the proof follows directly from Lemma 3.5.

Corollary 3.10. If $\tilde{z}_1$ and $\tilde{z}_2$ are rectangular L-fuzzy complex numbers, then so are $\tilde{z}_1 + \tilde{z}_2$, $\tilde{z}_1 - \tilde{z}_2$, $\tilde{z}_1 \cdot \tilde{z}_2$ and $\tilde{z}_1 \div \tilde{z}_2$, where in the case of division $0 \notin supp(\tilde{z}_2)$.

Proof. The proof follows from Theorem 3.9 and Theorem 2.6.

Theorem 3.11. If $\tilde{z}_1, \tilde{z}_2 \in RLF(\mathbb{C})$ and $\ell \in L$, then

(1) $(\tilde{z}_1 + \tilde{z}_2)_{(\ell)} = \tilde{z}_{1(\ell)} + \tilde{z}_{2(\ell)}$.

(2) $(\tilde{z}_1 + \tilde{z}_2\,)^{[\ell]} = \tilde{z}_1{}^{[\ell]} + \tilde{z}_2{}^{[\ell]}$.

Proof. (1) Suppose that $w \in \tilde{z}_{1(\ell)} + \tilde{z}_{2(\ell)}$, this implies there exists w_1 and w_2 such that $w_1 + w_2 = w$ and both belong to $\tilde{z}_{1(\ell)}$ and $\tilde{z}_{2(\ell)}$, respectively. So that, for $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, Definitions 3.1 and 3.2 conclude that

$$T_m\big(\mu(x_1|\tilde{x}_1), \mu(y_1|\tilde{y}_1)\big) > \ell \text{ and } T_m\big(\mu(x_2|\tilde{x}_2), \mu(y_2|\tilde{y}_2)\big) > \ell.$$

Next, the extension principle for L-fuzzy numbers, say that

$$\mu(x|\tilde{x}_1 + \tilde{x}_2) = \bigvee_{x=x_1+x_2} T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big) > \ell,$$

and

$$\mu(y|\tilde{y}_1 + \tilde{y}_2) = \bigvee_{y=y_1+y_2} T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big) > \ell.$$

Therefore, x and y belongs to $(\tilde{x}_1 + \tilde{x}_2)_{(\ell)}$ and $(\tilde{y}_1 + \tilde{y}_2)_{(\ell)}$ respectively. Finally, by Lemma 3.5 (3) and Theorem 3.7 (2) we get the result.

The proof of (2) is similar.

As we saw in Theorem 3.7, we have $\tilde{z}_{1[\ell]} + \tilde{z}_{2[\ell]} \subset (\tilde{z}_1 + \tilde{z}_2)_{[\ell]}$, to prove the other side we find the sufficient condition as follows:

Theorem 3.12. Let $\tilde{z}_1, \tilde{z}_2 \in RLF(\mathbb{C})$ and α is an $\vee - \wedge$ mapping. Then $(\tilde{z}_1 + \tilde{z}_2)_{[\ell]} = \tilde{z}_{1[\ell]} + \tilde{z}_{2[\ell]}$ for all $\ell \in L$.

Proof. For $w \in (\tilde{z}_1 + \tilde{z}_2)^{[\ell]}$ implies $\ell \notin \alpha(\mu(w|\tilde{z}_1 + \tilde{z}_2))$ and by Definition 3.3, we have that

$$\ell \notin \alpha\left(\bigvee_{w=w_1+w_2} T_m\big(\mu(w_1|\tilde{z}_1)\,,\mu(w_2|\tilde{z}_2)\big) \right).$$

Since α is an $\vee - \wedge$ mapping, then

$$\alpha\left(\bigvee_{w=w_1+w_2} T_m\big(\mu(w_1|\tilde{z}_1)\,,\mu(w_2|\tilde{z}_2)\big) \right) = \bigcap_{w=w_1+w_2} \alpha\left(T_m\big(\mu(w_1|\tilde{z}_1)\,,\mu(w_2|\tilde{z}_2)\big)\right)$$

Hence, for all w_1 and w_2 satisfying $w_1 + w_2 = w$, $\ell \notin \alpha\left(T_m\big(\mu(w_1|\tilde{z}_1)\,,\mu(w_2|\tilde{z}_2)\big)\right)$, and we know $\big(\alpha(\mu(w_1|\tilde{z}_1) \cup \alpha(\mu(w_2|\tilde{z}_2)\big) \subset \alpha\left(T_m\big(\mu(w_1|\tilde{z}_1)\,,\mu(w_2|\tilde{z}_2)\big)\right)$, therefore $\ell \notin \big(\alpha(\mu(w_1|\tilde{z}_1) \cup \alpha(\mu(w_2|\tilde{z}_2)\big)$ so that $\ell \notin \alpha(\mu(w_1|\tilde{z}_1)$ and $\ell \notin \alpha(\mu(w_1|\tilde{z}_1)$, for $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, Definitions 3.1 and 3.2 get that

$$\ell \notin \alpha\left(T_m\big(\mu(x_1|\tilde{x}_1), \mu(y_1|\tilde{y}_1)\big)\right) \text{and } \ell \notin \left(T_m\big(\mu(x_2|\tilde{x}_2), \mu(y_2|\tilde{y}_2)\big)\right)$$

In other words, $\ell \notin \alpha\big(\mu(x_1|\tilde{x}_1)\big)$, $\ell \notin \alpha\big(\mu(x_2|\tilde{x}_2)\big)$, $\ell \notin \alpha\big(\mu(y_1|\tilde{y}_1)\big)$ and $\ell \notin \alpha\big(\mu(y_2|\tilde{y}_2)\big)$, by definition of ℓ-cut of L-fuzzy numbers imply that

$$x_1 \in \tilde{x}_1{}^{[\ell]} \text{ and } x_2 \in \tilde{x}_2{}^{[\ell]},$$

also,

$$y_1 \in \tilde{y}_1{}^{[\ell]} \text{ and } y_2 \in \tilde{y}_2{}^{[\ell]}.$$

Hence, by Lemma 3.5 (1) we get that

$$w \in \left(\tilde{x}_1{}^{[\ell]} + i\tilde{y}_1{}^{[\ell]} + \tilde{x}_2{}^{[\ell]} + i\tilde{y}_2{}^{[\ell]}\right) = \tilde{z}_1{}^{[\ell]} + \tilde{z}_2{}^{[\ell]}.$$

By (2) in Theorem 3.7 we obtain the equality, as required.

As we saw in Theorem 3.7, we have $(\tilde{z}_1 + \tilde{z}_2\,)^{(\ell)} \subset \tilde{z}_1{}^{(\ell)} + \tilde{z}_2{}^{(\ell)}$, to prove the other side we find the sufficient condition as follows:

Theorem 3.13. Let $\tilde{z}_1, \tilde{z}_2 \in RLF(\mathbb{C})$. If for any $\ell_1, \ell_2 \in L$, $\beta(\ell_1 \wedge \ell_2) = \beta(\ell_1) \cap \beta(\ell_2)$, then $(\tilde{z}_1 + \tilde{z}_2\,)^{(\ell)} = \tilde{z}_1{}^{(\ell)} + \tilde{z}_2{}^{(\ell)}$ for all $\ell \in L$.

Proof. For $w \in \tilde{z}_1{}^{(\ell)} + \tilde{z}_2{}^{(\ell)}$, there exists w_1 and w_2 satisfying $w_1 + w_2 = w$ such that $w_1 \in \tilde{z}_1{}^{(\ell)}$ and $w_2 \in \tilde{z}_2{}^{(\ell)}$, which gets that $\ell \in \beta\big(\mu(w_1|\tilde{z}_1)\big)$ and $\ell \in \beta\big(\mu(w_2|\tilde{z}_2)\big)$. For $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, By hypothesis, ℓ belongs to all $\beta\big(\mu(x_1|\tilde{x}_1)\big)$, $\beta\big(\mu(x_2|\tilde{x}_2)\big)$, $\beta\big(\mu(y_1|\tilde{y}_1)\big)$ and $\beta\big(\mu(y_2|\tilde{y}_2)\big)$. One apply Theorem 2.4, we have that

$$\ell \in \beta\left(\bigvee_{x=x_1+x_2} T_m\big(\mu(x_1|\tilde{x}_1), \mu(x_2|\tilde{x}_2)\big)\right) \text{ and } \ell \in \beta\left(\bigvee_{y=y_1+y_2} T_m\big(\mu(y_1|\tilde{y}_1), \mu(y_2|\tilde{y}_2)\big)\right).$$

Therefore,

$$x \in (\tilde{x}_1 + \tilde{x}_2)^{(\ell)} \text{ and } y \in (\tilde{y}_1 + \tilde{y}_2)^{(\ell)}$$

By Lemma 3.6 we see that $w = x + iy \in (\tilde{z}_1 + \tilde{z}_2)^{(\ell)}$. Hence, the result follows from Theorem 3.7 (2).

Theorem 3.14. Let $\tilde{z}$ be a rectangular L-fuzzy complex number. If for any $\ell, \mu \in L$, $\ell \leq \mu$, then

(1) $\tilde{z}^{[\mu]} \subseteq \tilde{z}^{[\ell]}$.
(2) $\tilde{z}_{[\mu]} \subseteq \tilde{z}_{[\ell]}$.
(3) $\tilde{z}_{(\mu)} \subseteq \tilde{z}_{(\mu)}$.

Proof. (1). Suppose that $z \in \tilde{z}^{[\mu]}$. Then $\mu(z|\tilde{z}) \geq \mu$ and for $z = x + iy$. By Definition 3.1 we have $T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big) \geq \mu \geq \ell$ this implies that $\mu(x|\tilde{x}) \geq \ell$ and $\mu(y|\tilde{y}) \geq \ell$. So, by Lemma 3.5 (1) we get that $z \in \tilde{z}^{[\ell]}$.

(2). For $z \in \tilde{z}_{[\mu]}$ we have that $\mu \notin \alpha(\mu(z|\tilde{z})$. Then, $\mu \notin \alpha\left(T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big)\right)$. Next, Theorem 2.4 implies that $\mu \notin \left(\alpha\big(\mu(x|\tilde{x})\big) \cup \alpha\big(\mu(y|\tilde{y})\big)\right)$. So, it remains to show that $\mu \notin \alpha\big(\mu(x|\tilde{x})\big)$ and $\mu \notin \alpha\big(\mu(y|\tilde{y})\big)$. Since $\ell \leq \mu$, by Definition 2.2 it is obvious that $\ell \notin \alpha\big(\mu(x|\tilde{x})\big)$ and $\ell \notin \alpha\big(\mu(y|\tilde{y})\big)$. Therefore, by Definition 3.1 and Lemma 3.5 (2) we have $z \in \tilde{z}_{[\ell]}$.

The proof of (3) is similar as (1).

Theorem 3.15. Let $\tilde{z}$ be a rectangular L-fuzzy complex number. If for any $\ell, \mu \in L$, $\beta(\ell \wedge \mu) = \beta(\ell) \cap \beta(\mu)$, then $\tilde{z}^{(\mu)} \subseteq \tilde{z}^{(\ell)}$.

Proof. Assume that $z \in \tilde{z}^{(\mu)}$. Then $\mu \in \beta\big(\mu(z|\tilde{z})\big)$ and for $z = x + iy$, Definition 3.1 implies that $\mu \in \beta\left(T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big)\right)$. So, μ belongs to both $\beta\big(\mu(x|\tilde{x})\big)$ and $\beta\big(\mu(y|\tilde{y})\big)$. Since, $\ell \leq \mu$, then $\ell \in \beta\big(\mu(x|\tilde{x})\big)$ and $\ell \in \beta\big(\mu(y|\tilde{y})\big)$. From Lemma 3.6 we obtain $z \in \tilde{z}^{(\ell)}$.

Definition 3.16. Given $\tilde{z} = \tilde{x} + i\tilde{y}$, the modulus of $\tilde{z}$ define by $|\tilde{z}| = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, where $|\tilde{z}|$ is an L-fuzzy number with the following membership function

$$\mu(r||\tilde{z}|) = \bigvee_{r=\sqrt{x^2+y^2}} T_m\big(\mu(x|\tilde{x}), \mu(y|\tilde{y})\big).$$

Lemma 3.17. If $\tilde{z} = \tilde{x} + i\tilde{y}$ be a rectangular L-fuzzy complex number and $\ell \in L$, then

(1) $|\tilde{z}|^{[\ell]} = |\tilde{z}^{[\ell]}|$,
(2) $|\tilde{z}|^{(\ell)} = |\tilde{z}^{(\ell)}|$,
(3) $|\tilde{z}|_{(\ell)} = |\tilde{z}_{(\ell)}|$,
(4) $|\tilde{z}|_{[\ell]} = |\tilde{z}_{[\ell]}|$.

Proof. (1) Let $r \in |\tilde{z}\,|^{[\ell]}$. Then there exists $z = x + iy$ such that $r = \sqrt{x^2 + y^2}$ and $T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big) \geq \ell$. Hence $r \in |\tilde{z}^{\,[\ell]}|$. Conversely, let $r \in |\tilde{z}^{\,[\ell]}|$ then there are x and y so that $r = \sqrt{x^2 + y^2}$ and $T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big) \geq \ell$. This implies that $\bigvee_{r=\sqrt{x^2+y^2}} T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big) \geq \ell$. Therefore, $r \in |\tilde{z}\,|^{[\ell]}$.

(2) Suppose that $r \in |\tilde{z}\,|^{(\ell)}$. So, by Theorem 2.4, there exists $z = x + iy$ such that $r = \sqrt{x^2 + y^2}$ and $\ell \in \beta\Big(T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big)\Big)$. This means that $r \in |\tilde{z}^{\,(\ell)}|$. Conversely, let $r \in |\tilde{z}^{\,(\ell)}|$, then there are x and y so that $r = \sqrt{x^2 + y^2}$ and $\ell \in \beta\Big(T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big)\Big)$, clear that $\ell \in \beta\Big(\bigvee_{r=\sqrt{x^2+y^2}} T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big)\Big)$ and this implies that $r \in |\tilde{z}\,|^{(\ell)}$.

The proof of (3) and (4) follows by similar steps.

Definition 3.18. If $\tilde{z} = \tilde{x} + i\tilde{y}$ is any rectangular L-fuzzy complex number, then the conjugate of $\tilde{z}$ define by $\bar{\tilde{z}} = \tilde{x} + i(-\tilde{y})$ with the following membership function

$$\mu(z|\bar{\tilde{z}}\,) = \bigvee_{z=x-iy} T_m\big(\mu(x|\tilde{x}\,), \mu(y|\tilde{y}\,)\big)$$

Theorem 3.19. For a rectangular L-fuzzy complex number $\tilde{z} = \tilde{x} + i\tilde{y}$ and $\ell \in L$, we have

(1) $\bar{\tilde{z}}^{\,[\ell]} = \overline{\tilde{z}^{\,[\ell]}}$
(2) $\bar{\tilde{z}}_{\,[\ell]} = \overline{\tilde{z}_{\,[\ell]}}$
(3) $\bar{\tilde{z}}_{\,(\ell)} = \overline{\tilde{z}_{\,(\ell)}}$
(4) $\bar{\tilde{z}}^{\,(\ell)} = \overline{\tilde{z}^{\,(\ell)}}$

Proof. The proof follows from Lemma 3.5 (1-3), Lemma 3.6, and Definition 3.17.

L-multi-fuzzy complex numbers

Let p be the set of positive integers and $\{L_i : i \in p\}$ a family of complete lattices. A collection of repeated symbols is called multiset [17] and fuzzy multiset [6] defined as a fuzzy bag $\tilde{\gamma}$ drawn from A characterized by a function $\mu_{\tilde{\gamma}} : A \to \gamma$, where γ is the set of all classical bags drawn from $[0, 1]$. Next, Sebastian and Ramakrishnan [18] developed the concept of fuzzy multisets in terms of ordered sequences of membership functions as a set $\widetilde{\Pi} = \{(a,\ \mu_1(a), \mu_2(a), \dots, \mu_i(a), \dots) : a \in A\}$, where $\mu_i \in L_i(A)$ for $i \in p$. Also, Atanassov's in [5] defined intuitionistic fuzzy set $\widetilde{\Omega}$ in the universal set X as an object of the form $\widetilde{\Omega} = \{(x, \mu_{\widetilde{\Omega}}(x), \nu_{\widetilde{\Omega}}(x)) : x \in X\}$, where $\mu_{\widetilde{\Omega}}(x)$ (resp. $\nu_{\Omega}(x)$) define the degree of membership (resp. the degree of non-membership) of elements $x \in X$ to the fuzzy set $\widetilde{\Omega}$ in X, and for every $x \in X$, $0 \leq \mu_{\widetilde{\Omega}}(x) + \nu_{\widetilde{\Omega}}(x) \leq 1$.

Let X be the universal set, $\mathbb{N}^0$ the set of nonnegative integers and let $Count_A : X \to \mathbb{N}^0$ be a function. We define a L-fuzzy multiset $\tilde{x}$ in X by a multiset of $X \times L$ as $\tilde{x} = \{Count_{\tilde{x}}(x, \mu(x|\tilde{x}))/(x, \mu(x|\tilde{x})) : x \in X, \mu \in L\}$, that is, the number of x with the membership grade μ is $Count_{\tilde{x}}(x, \mu)$. We also can write a decreasing sequence of values of $x \in X$ as $\tilde{x} = \big\{\big(x, \mu_1(x|\tilde{x}), \mu_2(x|\tilde{x}), \dots, \mu_k(x|\tilde{x})\big) : x \in X\big\}$, where $\mu_i \in L_i(X)$ for $i = 1, 2, \dots, k$, $k \in p$ is independent of x and $\mu_1(x|\tilde{x}) \geq \mu_2(x|\tilde{x}) \geq \cdots \geq \mu_k(x|\tilde{x})$. For $a \in L$, the copies number of $x \in \tilde{x}(a)$, the a-cut of L-fuzzy multiset $\tilde{x}$, is defined as $Count_{\tilde{x}(a)}(x) = 0$ if and only if $\mu_1(x|\tilde{x}) < a$; $Count_{\tilde{x}(a)}(x) = p$ if and only if $\mu_p(x|\tilde{x}) \geq a$ and $\mu_{p+1}(x|\tilde{x}) < a$ for $p < k$; $Count_{\tilde{x}(a)}(x) = k$ if and only if $\mu_k(x|\tilde{x}) \geq a$.

An L-fuzzy multiset $\tilde{x} = \{(x, \mu_1(x|\tilde{x}), \mu_2(x|\tilde{x}), \dots, \mu_k(x|\tilde{x})) : x \in X, \mu_i \in L_i(\mathbb{R}), i = 1,2, \dots, k\}$ is called L-multi-fuzzy number if

(1) There exists $x \in \mathbb{R}$ such that $\mu_i(x|\tilde{x}) = 1$ for $i = 1,2, \dots, k$.
(2) For all $a \in L$, $\tilde{x}(a)$ is a bounded closed interval.

If $\tilde{x}$ and $\tilde{y}$ are two L-multi-fuzzy numbers with membership functions $\mu_k(x|\tilde{x})$ and $\mu_k(y|\tilde{y})$, respectively, then $\tilde{z} = \tilde{x} + i\tilde{y}$ is a rectangular L-multi-fuzzy complex number with multi membership functions

$$\mu_k(x + iy|\tilde{z}) = T_m(\mu_k(x|\tilde{x}), \mu_k(y|\tilde{y})), k \in p.$$

For $a \in L$, the a-cut, $\tilde{z}(a)$, of L-multi-fuzzy complex number $\tilde{z} = \{(z, \mu_1(z|\tilde{z}), \mu_2(z|\tilde{z}), \dots, \mu_k(z|\tilde{z})) : z \in \mathbb{C}, \mu_i \in L_i(\mathbb{C}), i = 1,2, \dots, k\}$ is defined as:

$Count_{\tilde{z}(a)}(z) = 0$ if and only if $\mu_1(z|\tilde{z}) < a$;

$Count_{\tilde{z}(a)}(z) = p$ if and only if $\mu_p(z|\tilde{z}) \geq a$ and $\mu_{p+1}(z|\tilde{z}) < a$ for $p < k$;

$Count_{\tilde{z}(a)}(z) = k$ if and only if $\mu_k(z|\tilde{z}) \geq a$.

Lemma 4.1. If $\tilde{z} = \tilde{x} + i\tilde{y}$ is a rectangular L-multi-fuzzy complex number and $a \in L$, then

$$\tilde{z}(a) = \tilde{x}(a) + i\,\tilde{y}(a).$$

Proof. Let $z \in \tilde{z}(a)$ implies there exists a positive integer p whenever $Count_{\tilde{z}(a)}(z) = p$, if and only if $\mu_p(z|\tilde{z}) \geq a$, so that there is an x and y such that $x + iy = z$ and $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a$, it is obvious that $\mu_p(x|\tilde{x}) \geq a$ and $\mu_p(y|\tilde{y}) \geq a$. Moreover, by definition of a-cuts of L-multi-fuzzy numbers, $Count_{\tilde{x}(a)}(x) = p$ and $Count_{\tilde{y}(a)}(y) = p$, this is to say $x \in \tilde{x}(a)$, and $y \in \tilde{y}(a)$. Therefore, $z \in \tilde{x}(a) + i\,\tilde{y}(a)$. Conversely, if $z \in \tilde{x}(a) + i\,\tilde{y}(a)$, where $z = x + iy$, then obviously there is a positive integer p such that $Count_{\tilde{x}(a)}(x) = p$ and $Count_{\tilde{y}(a)}(y) = p$ if only if $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a$, clear that $\mu_p(z|\tilde{z}) \geq a$ implies $Count_{\tilde{z}(a)}(z) = p$. Therefore, we have shown that $z \in \tilde{z}(a)$.

Theorem 4.2. Let $\tilde{z}$ is a rectangular L-multi-fuzzy complex number. If for any $a, b \in L$, $a \leq b$, then

$$\tilde{z}(b) \subseteq \tilde{z}(a).$$

Proof. Suppose that $z \in \tilde{z}(b)$, then $Count_{\tilde{z}(b)}(z) = p$ if and only if $\mu_p(z|\tilde{z}) \geq b$, so that there is an x and y such that $x + iy = z$ and $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq b$, this means that $\mu_p(x|\tilde{x}) \geq b$ and $\mu_p(y|\tilde{y}) \geq b$. Since $a \leq b$, we get that $\mu_p(x|\tilde{x}) \geq a$ and $\mu_p(y|\tilde{y}) \geq a$. Moreover, $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a$ implies that $\mu_p(z|\tilde{z}) \geq a$ so that $Count_{\tilde{z}(a)}(z) = p$. This means that $z \in \tilde{z}(a)$.

Theorem 4.3. If $\tilde{z}$ is a rectangular L-multi-fuzzy complex number and $a \in L$, then

$$\tilde{z}(a) = \bigcap_{b<a} \tilde{z}(b).$$

Proof. Suppose $b < a$ so that by Theorem 4.2 we have $\tilde{z}(a) \subseteq \tilde{z}(b)$, hence,

$$\tilde{z}(a) \subseteq \bigcap_{b<a} \tilde{z}(b).$$

Conversely, for all $z \in \bigcap_{b<a} \tilde{z}(b)$ and every $\epsilon > 0$ we get $z \in \tilde{z}(a-\epsilon)$. Which includes that there is a positive integer p such that $Count_{\tilde{z}(a-\epsilon)}(z) = p$, this implies that $\mu_p(z|\tilde{z}) \geq a - \epsilon$, that is $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a - \epsilon$ since ϵ is any arbitrary positive number, suppose $\epsilon \to 0$. So, we conclude that $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a$ if and only if $Count_{\tilde{z}(a)}(z) = p$ for a positive integer p and $z \in \tilde{z}(a)$. Hence,

$$\bigcap_{b<a} \tilde{z}(b) \subseteq \tilde{z}(a).$$

Definition 4.4. If $\tilde{z}_1$ and $\tilde{z}_2$ are rectangular L-multi-fuzzy complex numbers and $*$ denotes the basic arithmetic operations, then we define the membership function of $\tilde{z}_1 * \tilde{z}_2$ as follows

$$\mu_k(w \mid \tilde{z}_1 * \tilde{z}_2) = \bigvee_{w=w_1*w_2} T_m\left(\mu_k(w_1|\tilde{z}_1), \mu_k(w_2|\tilde{z}_2)\right).$$

Theorem 4.5. If $\tilde{z}_1$ and $\tilde{z}_2$ are rectangular L-multi-fuzzy complex numbers, then for all $a \in L$,

$$(\tilde{z}_1 * \tilde{z}_2)(a) = \tilde{z}_1(a) * \tilde{z}_2(a)$$

Proof. We prove only for the case that $*$ is addition, the other operations can be proved similarly.

Let $w \in (\tilde{z}_1 + \tilde{z}_2)(a)$, then $Count_{(\tilde{z}_1+\tilde{z}_2)(a)}(w) = p$ if and only if $\mu_p(w|\tilde{z}_1 + \tilde{z}_2) \geq a$. By Definition 4.4 we get

$$\bigvee_{w=w_1+w_2} T_m\left(\mu_p(w_1|\tilde{z}_1), \mu_p(w_2|\tilde{z}_2)\right) \geq a$$

It follows that both $\mu_p(w_1|\tilde{z}_1)$ and $\mu_p(w_2|\tilde{z}_2)$ are greater than or equal to a. So that, for $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, the definition of rectangular L-multi-fuzzy numbers gets that

$$T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(y_1|\tilde{y}_1)\right) \geq a \text{ and } T_m\left(\mu_p(x_2|\tilde{x}_2), \mu_p(y_2|\tilde{y}_2)\right) \geq a.$$

On the other hand, we conclude that all $\mu_p(x_1|\tilde{x}_1)$, $\mu_p(y_1|\tilde{y}_1)$, $\mu_p(x_2|\tilde{x}_2)$ and $\mu_p(y_2|\tilde{y}_2)$ are greater than or equal to a if and only if $Count_{\tilde{x}_1(a)}(x_1) = p$, $Count_{\tilde{x}_2(a)}(x_2) = p$, $Count_{\tilde{y}_1(a)}(y_1) = p$ and $Count_{\tilde{y}_2(a)}(y_2) = p$. Then the definition of a-cut of L-multi-fuzzy numbers implies that $x_1 \in \tilde{x}_1(a), x_2 \in \tilde{x}_2(a), y_1 \in \tilde{y}_1(a)$ and $y_2 \in \tilde{y}_2(a)$. Hence, $w_1 \in \tilde{x}_1(a) + i\tilde{y}_1(a)$ and $w_2 \in \tilde{x}_2(a) + i\tilde{y}_2(a)$. It follows that $w \in \tilde{z}_1(a) + \tilde{z}_2(a)$ by Lemma 4.1.

Conversely, suppose that $w \in \tilde{z}_1(a) + \tilde{z}_2(a)$, then there exist w_1 and w_2 satisfying $w_1 + w_2 = w$ such that $w_1 \in \tilde{z}_1(a)$ and $w_2 \in \tilde{z}_1(a)$, this illustrates that there exists a positive integer p such that $Count_{\tilde{z}_1(a)}(w_1) = p$

and $Count_{\tilde{z}_2(a)}\,(w_2) = p$ if and only if $\mu_p(w_1|\tilde{z}_1)\ \geq a$ and $\mu_p(w_2|\tilde{z}_2) \geq p$. Next, for $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$, the definition of L-multi-fuzzy numbers say that

$$T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(y_1|\tilde{y}_1)\right) \geq a \text{ and } T_m\left(\mu_p(x_2|\tilde{x}_2), \mu_p(y_2|\tilde{y}_2)\right) \geq a.$$

So that, the extension principle for L-multi-fuzzy numbers, gets that

$$\mu_p(x|\tilde{x}_1 + \tilde{x}_2) = \bigvee_{x=x_1+x_2} T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2)\right) \geq a,$$

and

$$\mu_p(y|\tilde{y}_1 + \tilde{y}_2) = \bigvee_{y=y_1+y_2} T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2)\right) \geq a.$$

Therefore, $Count_{(\tilde{x}_1+\tilde{x}_2)(a)}(x) = p$ and $Count_{(\tilde{x}_1+\tilde{x}_2)(a)}(x) = p$, these means that x and y belong to $(\tilde{x}_1 + \tilde{x}_2)(a)$ and $(\tilde{y}_1 + \tilde{y}_2)(a)$, respectively. Hence, $w \in \tilde{z}_1(a) + \tilde{z}_1(a)$ and the result follows by Lemma 4.1.

Theorem 4.6. If $\tilde{z}_1 = \tilde{x}_1 + i\tilde{y}_1$ and $\tilde{z}_2 = \tilde{x}_2 + i\tilde{y}_2$ are two rectangular L-multi-fuzzy complex numbers, then

(1) $\tilde{z}_1 + \tilde{z}_2 = \tilde{x}_1 + \tilde{x}_2 + i(\tilde{y}_1 + \tilde{y})$,
(2) $\tilde{z}_1 - \tilde{z}_2 = \tilde{x}_1 - \tilde{x}_2 + i(\tilde{y}_1 - \tilde{y})$,
(3) $\tilde{z}_1 \cdot \tilde{z}_2 = (\tilde{x}_1\tilde{x}_2 - \tilde{y}_1\tilde{y}_2) + i(\tilde{y}_1\tilde{x}_2 + \tilde{y}_2\tilde{x}_1)$,
(4) $\frac{\tilde{z}_1}{\tilde{z}_2} = \left(\frac{\tilde{x}_1\tilde{x}_2+\tilde{y}_1\tilde{y}_2}{\tilde{x}_2^2+\tilde{y}_2^2}\right) + i\left(\frac{\tilde{y}_1\tilde{x}_2+\tilde{y}_2\tilde{x}_1}{\tilde{x}_2^2+\tilde{y}_2^2}\right)$, where $0 \notin supp(\tilde{x}_2^2 + \tilde{y}_2^2)$.

Proof. (1) It is obvious that for $w_1 = x_1 + iy_1$ and $w_2 = x_2 + iy_2$ we have $T_m\left(\mu_p(w_1|\tilde{z}_1), \mu_p(w_2|\tilde{z}_2)\right) = T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2), \mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right)$. Moreover, let $\tilde{x}$, $\tilde{y}$, x, and y denote for $\tilde{x}_1 + \tilde{x}_2$, $\tilde{y}_1 + \tilde{y}_2$, $x_1 + x_2$ and $y_1 + y_2$ respectively, then

$$T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2), \mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right) \leq T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2)\right)$$

and

$$T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2), \mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right) \leq T_m\left(\mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right)$$

This means that, $T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2), \mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right)$ less than or equal to both $\mu_p(x\,|\tilde{x})$ and $\mu_p(y\,|\tilde{y})$. Moreover, for $\tilde{z} = \tilde{x} + i\tilde{y}$ and $w = x + iy$, $T_m\left(\mu_p(x_1|\tilde{x}_1), \mu_p(x_2|\tilde{x}_2), \mu_p(y_1|\tilde{y}_1), \mu_p(y_2|\tilde{y}_2)\right) \leq \mu_p(w\,|\tilde{z})$ means that $\mu_p(w \mid \tilde{z}_1 + \tilde{z}_2) \leq \mu_p(z\,|\tilde{z})$.

On the other hand, for arbitrary $\epsilon > 0$ there are x_1', y_1', x_2' and y_2' such that $x = x_1' + x_2'$, $y = y_1' + y_2'$ and $T_m\left(\mu_p(x_1'|\tilde{x}_1), \mu_p(x_2'|\tilde{x}_2)\right) > \mu_p(x\,|\tilde{x}) - \epsilon \left(resp.\, T_m\left(\mu_p(y_1'|\tilde{y}_1), \mu_p(y_2'|\tilde{y}_2)\right) > \mu_p(y\,|\tilde{y}) - \epsilon\right)$. This implies that,

$$T_m\left(\mu_p(x_1'|\tilde{x}_1), \mu_p(x_2'|\tilde{x}_2), \mu_p(y_1'|\tilde{y}_1), \mu_p(y_2'|\tilde{y}_2)\right) > \mu_p(w|\tilde{z}) - \epsilon$$

So that, $\mu_p(w \mid \tilde{z}_1 + \tilde{z}_2) > \mu_p(w|\tilde{z}) - \epsilon$ implies that $\mu_p(w \mid \tilde{z}_1 + \tilde{z}_2) = \mu_p(w|\tilde{z})$.

The proof of (2), (3), and (4) are similar to (1).

Lemma 4.7. If $\tilde{x}$ and $\tilde{y}$ are two L-multi-fuzzy numbers and $*$ denote the basic arithmetic operations, then so are $\tilde{x} * \tilde{y}$. Where in the case of division $Count_{\tilde{y}(a)}(0) \neq 0$, for all $a \in L$.

Proof. We only prove for the case that $*$ is addition. The other cases are similar.

First, for $\widetilde{w} = \tilde{x} + \tilde{y}$ and $a \in L$, since both $\tilde{x}$ and $\tilde{y}$ are L-multi-fuzzy numbers, clear that $\tilde{x}$ and $\tilde{y}$ are normal. So that, there exist x and y such that $Count_{\tilde{x}(1)}(x) = p$ and $Count_{\tilde{y}(1)}(y) = p$. Moreover, if $z = x + y$ then $Count_{\tilde{x}+\tilde{y}(1)}(z) = p$ and this implies that $\widetilde{w}$ is nonempty.

Second, to show $\widetilde{w}(a)$ is closed. Suppose that $\{w_n\}$ is a sequence in $\widetilde{w}(a)$ such that $w_n = x_{1n} + x_{2n}$, where $\{x_{1n}\}$ and $\{x_{2n}\}$ are sequences in $\tilde{x}(a)$ and $\tilde{y}(a)$, respectively. So obviously x_{1n} and x_{2n} are convergent sequences in them. Moreover, if x_{1n} converges to x and x_{2n} converges to y such that $x \in \tilde{x}(a)$ and $y \in \tilde{y}(a)$ then w_n also converges to $x + y$. Finally, since $\tilde{x}$ and $\tilde{y}$ are L-multi-fuzzy numbers so that $\tilde{x}(a)$ and $\tilde{y}(a)$ are bounded sets, this implies that $\widetilde{w}(a) = (\tilde{x} + \tilde{y})(a) = \tilde{x}(a) + \tilde{y}(a)$ also is bounded for all $a \in L$.

Theorem 4.8. If $\tilde{z}$ and $\widetilde{w}$ are two L-multi-fuzzy complex numbers and $*$ denotes the basic arithmetic operations, then so are $\tilde{z} * \widetilde{w}$. Where in the case of division $Count_{\widetilde{w}(a)}(0) \neq 0$, for all $a \in L$.

Proof. The proof follows from Theorem 4.6 and Lemma 4.7.

Definition 4.9. If $\tilde{z}$ is a rectangular L-multi-fuzzy complex number, then the modulus of $\tilde{z}$ is defined by $|\tilde{z}| = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, where $|\tilde{z}|$ is an L-multi-fuzzy number with the following membership function

$$\mu_p(r||\tilde{z}|) = \bigvee_{r=\sqrt{x^2+y^2}} T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right).$$

Lemma 4.10. If $\tilde{z}$ is a rectangular L-multi-fuzzy complex number and $a \in L$, then

$$|\tilde{z}|(a) = |\tilde{z}(a)|.$$

Proof. Suppose $r \in |\tilde{z}|(a)$, so there exists $z = x + iy$ such that $r = \sqrt{x^2 + y^2}$ and $Count_{|\tilde{z}|(a)}(r) = p$ if and only if $\mu_p(r||\tilde{z}|) \geq a$, implies that

$$\bigvee_{r=\sqrt{x^2+y^2}} T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a.$$

Finally, $T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right) \geq a$ if and only if $Count_{|\tilde{z}(a)|}(r) = p$.

Theorem 4.11. If $\tilde{z}_1$ and $\tilde{z}_2$ are two rectangular L-multi-fuzzy complex numbers, then

(1) $|\tilde{z}_1 + \tilde{z}_2| \leq |\tilde{z}_1| + |\tilde{z}_2|$.
(2) $|\tilde{z}_1 - \tilde{z}_2| \geq |\tilde{z}_1| - |\tilde{z}_2|$.

Proof. We prove only (1). Lemma 4.10 and Theorem 4.5 implies that

$$|\tilde{z}_1 + \tilde{z}_2|(a) = |(\tilde{z}_1 + \tilde{z}_2)(a)| = |\tilde{z}_1(a) + \tilde{z}_2(a)|$$

$$
\begin{aligned}
&= \{|z_1 + z_2| \,|\, z_1 \in \tilde{z}_1(a), z_2 \in \tilde{z}_2(a)\} \\
&= \{|z_1 + z_2| \,|\, Count_{\tilde{z}_1(a)}(z_1) = p, Count_{\tilde{z}_2(a)}(z_2) = p\} \\
&\leq \{|z_1| + |z_2| \,|\, Count_{\tilde{z}_1(a)}(z_1) = p, Count_{\tilde{z}_2(a)}(z_2) = p\} \\
&= |\tilde{z}_1(a)| + |\tilde{z}_2(a)| = |\tilde{z}_1|(a) + |\tilde{z}_2|(a) \\
&= (|\tilde{z}_1| + |\tilde{z}_2|)(a)
\end{aligned}
$$

Definition 4.12. Given $\tilde{z} = \tilde{x} + i\tilde{y}$ is any rectangular L-multi-fuzzy complex number, then the conjugate of $\tilde{z}$ is defined by $\bar{\tilde{z}} = \tilde{x} + i(-\tilde{y})$. where $\tilde{z}$ also is a new L-multi-fuzzy complex number with the following membership function

$$\mu_p(z|\bar{\tilde{z}}) = \bigvee_{z=x-iy} T_m\left(\mu_p(x|\tilde{x}), \mu_p(y|\tilde{y})\right).$$

Lemma 4.13. For a rectangular L-multi-fuzzy complex number $\tilde{z} = \tilde{x} + i\tilde{y}$ and $a \in L$, we have

$$\bar{\tilde{z}}(a) = \overline{\tilde{z}(a)}$$

Proof. The result follows from Lemma 4.1 and Definition 4.10.

Theorem 4.14. For a rectangular L-multi-fuzzy complex number $\tilde{z}$, we have $\bar{\bar{\tilde{z}}} = \tilde{z}$.

Proof: In view of Lemma 4.13, we can see that $\bar{\bar{\tilde{z}}}(a) = \overline{\bar{\tilde{z}}(a)} = \overline{\overline{\tilde{z}(a)}} = \{\bar{\bar{z}} \,|z \in \tilde{z}(a)\} = \tilde{z}(a)$ for all $a \in L$.

Theorem 4.15. If $\tilde{z}_1$ and $\tilde{z}_2$ are rectangular L-multi-fuzzy complex numbers and $*$ denote the basic arithmetic operation, then

$$\overline{(\tilde{z}_1 * \tilde{z}_2)} = \bar{\tilde{z}}_1 * \bar{\tilde{z}}_2.$$

Proof. For any $a \in L$, we have

$$
\begin{aligned}
\overline{(\tilde{z}_1 * \tilde{z}_2)}(a) = \overline{(\tilde{z}_1 * \tilde{z}_2)(a)} &= \overline{\left(\tilde{z}_1(a) * \tilde{z}_2(a)\right)} \\
&= \{\, \overline{z_1 * z_2} \,|\, z_1 \in \tilde{z}_1(a), z_2 \in \tilde{z}_2(a)\} \\
&= \{\, \overline{z_1 * z_2} \,|\, Count_{\tilde{z}_1(a)}(z_1) = p, Count_{\tilde{z}_2(a)}(z_2) = p\} \\
&= \{\overline{z_1} * \overline{z_1} | \, Count_{\tilde{z}_1(a)}(z_1) = p, Count_{\tilde{z}_2(a)}(z_2) = p\} \\
&= \overline{\tilde{z}_1(a)} * \overline{\tilde{z}_1(a)} = \bar{\tilde{z}}_1(a) * \bar{\tilde{z}}_2(a) = (\bar{\tilde{z}}_1 * \bar{\tilde{z}}_2)(a).
\end{aligned}
$$

Hence, Lemma 4.13 follows the result.

Conclusions

In this paper, the concept of $[0, 1]$-fuzzy complex number is generalized to a rectangular L-fuzzy complex number and L-multi-fuzzy complex number. Some of their characterizations and properties are presented by using the lower cut sets and the upper cut sets of L-fuzzy set theory and some decompositions of them are obtained. Also, we showed that there is a relationship between the rectangular L-fuzzy complex numbers and L-fuzzy numbers under the union and intersection preserving mapping conditions. Furthermore, we proved that L-multi-fuzzy complex numbers can be characterized by their rectangular L-fuzzy complex numbers.

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